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Lagrangian submanifolds and higher-order mechanical systems

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Abstract. Higher-order Lagrangian and Hamiltonian systems (time dependent or independent) are interpreted in terms of Lagrangian submanifolds of symplectic higher-order tangent bundles. The relation between both formalisms is given.

1. Introduction

In a previous paper (de León and Lacomba 1988), we saw that, given any higher-order Lagrangian function not necessarily regular, we can define a Lagrangian submanifold of a symplectic higher-order tangent bundle, which in local coordinates gives the Euler–Lagrange equations. This was done by using Tulczyjew's (1975, 1976a) notion of special symplectic manifolds and their Lagrangian submanifolds given by generating functions.

In this paper we complete the picture by showing how to relate with other Lagrangian submanifolds generated by the associated energy and the Hamiltonian in case the Lagrangian is regular. The higher-order case is more complicated than the first-order case, as we will see in § 3. An extension is also made to the time-dependent case in § 4 for the first-order case, and in § 5 for the higher-order case.

2. First-order Lagrangian and Hamiltonian dynamics.

All manifolds and maps are here supposed to be C^∞ differentiable. Given a manifold M, one considers new manifolds known as the tangent bundle TM and the cotangent bundle T^*M of M. In general we denote by τ_M and π_M, respectively, the canonical projections of the tangent bundle TM and the cotangent bundle T^*M on the manifold M, while θ_M denotes the Liouville form on T^*M defined by

$$\langle u, \theta_M(p)\rangle = \langle T\pi_M(u), p\rangle \tag{1}$$

for any $u \in T_pT^*M$, $p \in T^*M$.

Definition (Tulczyjew 1976a). A *special symplectic manifold* is a quintuple (X, M, π, λ, A) where $\pi: X \to M$ is a fibre bundle, λ is a 1-form on X and $A: X \to T^*M$ is a diffeomorphism such that $\pi = \pi_M \circ A$ and $\lambda = A^*\theta_M$. If K is a submanifold of M and $L: K \to R$ is a function, the *Lagrangian submanifold generated* by L in the symplectic manifold $(X, \mathrm{d}\lambda)$ is defined as

$$N = \{p \in X / \pi(p) \in K, \langle u, \lambda\rangle = \langle T\pi(u), \mathrm{d}L\rangle,$$
$$\text{for any } u \in TX \text{ such that } \tau_X(u) = p \text{ and } T\pi(u) \in TK \subset TM\}. \tag{2}$$

If $i: K \to M$ is an embedding and $L: K \to R$, we may identify K with $i(K) \subset M$ and generate the Lagrangian submanifold N according to this definition. We denote by $i^*: T^*M/K \to T^*K$ the mapping defined as follows: if $x \in K$, then $i^*: T^*_x M \to T^*_x K$ is given by $i^*\alpha = \alpha \circ Ti$ for all $\alpha \in T^*_x M$. For the sake of simplicity we shall omit the subscript and use the notation $i^*: T^*M \to T^*K$.

Proposition. If $i: K \to M$ is an embedding, then

$$i^* \circ A(N) = \mathrm{d}L(K).$$

Proof. We assume without loss of generality the identification $K \simeq i(K) \subset M$. If $p \in N$ we have that $k = \pi(p) \in K$ and $\langle u_p \, , A^*\theta_M \rangle = \langle T\pi(u_p), \mathrm{d}L(k) \rangle$ for any $u_p \in T_pX$ such that $T\pi(u_p) \in TK$. But

$$\langle u_p, A^*\theta_M(p) \rangle = \langle TA(u_p), \theta_M(A(p)) \rangle = \langle T\pi(u_p), i^* \circ A(p) \rangle$$

for any $T\pi(u_p)$, so $i^* \circ A(p) = \mathrm{d}L(k)$. We conclude the proof with the remark that $\pi(N) = K$. In fact, it is clear that $\pi(N) \subset K$. To prove the other inclusion we proceed as follows. Let $k \in K$; then there exists an element $z \in T^*_k M$ such that $i^*(z) = \mathrm{d}L(k)$. We set $p = A^{-1}(z)$. Thus, we have

$$\begin{aligned} \langle u_p, \lambda(p) \rangle &= \langle u_p, A^*\theta_M(p) \rangle = \langle TA(u_p), \theta_M(A(p)) \rangle \\ &= \langle TA(u_p), \theta_M(z) \rangle = \langle T\pi_M \circ TA(u_p), z \rangle \\ &= \langle T\pi(u_p), \mathrm{d}L(k) \rangle \end{aligned}$$

for all $u_p \in T_pX$ such that $T\pi(u_p) \in TK$. Since $\pi(p) = \pi \circ A^{-1}(z) = \pi_M(z) = k \in K$, we deduce that $p \in N$. Then $K \subset \pi(N)$.

Let V be an arbitrary manifold. We can define a canonical diffeomorphism $B_V: TT^*V \to T^*T^*V$ satisfying $\pi_{T^*V} \circ B_V = \tau_{T^*V}$ as follows. Given $v \in TT^*V$ we define $B_V(v) \in T^*T^*V$ by its action on any $w \in TT^*V$ such that $\tau_{T^*V}(v) = \tau_{T^*V}(w)$ as

$$\langle w, B_V(v) \rangle = (\mathrm{d}\theta_V)(v, w).$$

One verifies that in local coordinates (q^i) for V, (q^i, p_i) for T^*V and $(q^i, p_i, \dot{q}^i, \dot{p}_i)$ for TT^*V we have

$$B_V(q^i, p_i, \dot{q}^i, \dot{p}_i) = (q^i, p_i, \dot{p}_i, -\dot{q}^i).$$

In fact, B_V is nothing but the diffeomorphism defined by the canonical symplectic structure $\omega_V = \mathrm{d}\theta_V$ on T^*V, i.e. $B_V(v) = i(v)\omega_V$ (see Godbillon 1969). Define a 1-form β_V on TT^*V by

$$\beta_V = B^*_V(\theta_{T^*V}).$$

Locally we have

$$\beta_V = \dot{p}_i \, \mathrm{d}q^i - \dot{q}^i \, \mathrm{d}p_i.$$

Then $(TT^*V, T^*V, \tau_{T^*V}, \beta_V, B_V)$ is a special symplectic manifold. Consider the case $V = Q$, $K = M = T^*Q$ and let $H: T^*Q \to R$ be a Hamiltonian function on T^*Q. Then the corresponding Hamiltonian vector field ξ_H is defined as the section $\xi_H = B_Q^{-1} \circ \mathrm{d}(-H)$ or, equivalently,

$$i(\xi_H)\omega_Q = -\mathrm{d}H.$$

Its image is a Lagrangian submanifold in the symplectic manifold $(TT^*Q, \mathrm{d}\beta_Q)$. Locally we have

$$\xi_H(q^i, p_i) = (q^i, p_i, (\partial H/\partial p_i), -(\partial H/\partial q^i)).$$

Then the integral curves of ξ_H satisfy the *Hamilton equations* for H:

$$\dot{q}^i = (\partial H/\partial p_i) \qquad \dot{p}_i = -(\partial H/\partial q^i). \tag{3}$$

Consider the above construction for $K = M = T^*Q$, $X = TT^*Q$. Then we can check that the Lagrangian submanifold N_1 generated by $-H$ is the image of ξ_H, i.e. $N_1 = \xi_H(T^*Q)$ by taking into account (1). The following diagram illustrates the above situation:

$$\begin{tikzcd}
N_1 \arrow[d, hook] & & \\
TT^*Q \arrow[rr, "\beta_Q"] \arrow[drr, shift right, "\tau_{T^*Q}"'] & & T^*T^*Q \arrow[d, shift right, "\pi_{T^*Q}"'] \\
& & T^*Q \arrow[ull, shift right, "\xi_H"'] \arrow[u, shift right, "-\mathrm{d}H"'] \arrow[r, "H"] & R.
\end{tikzcd}$$

Now, we can define a canonical diffeomorphism $A_V: TT^*V \to T^*TV$. First, let us recall that there exists a canonical involution $S_V: TTV \to TTV$ locally given by

$$S_V(q^i, \dot{q}^i, \delta q^i, \delta\dot{q}^i) = (q^i, \delta q^i, \dot{q}^i, \delta\dot{q}^i)$$

where $(q^i, \dot{q}^i, \delta q^i, \delta\dot{q}^i)$ are local coordinates for TTV (see Godbillon 1969, Tulczyjew 1976). Now, given $v \in TT^*V$ we must define $A_V(v) \in T^*TV$ by means of its pairing on any element $w \in TTV$, where $\tau_{TV}(w) = T\pi_V(v)$. Given two curves $\gamma: R \to TV$ and $\chi: R \to T^*V$ such that $j_0^1\gamma = S_V(w)$, $j_0^1\chi = v$, and $\tau_V \circ \gamma = \pi_V \circ \chi$, we define

$$\langle w, A_V(v)\rangle = (\mathrm{d}/\mathrm{d}t)\langle \gamma, \chi\rangle(0).$$

In local coordinates we have

$$A_V(q^i, p_i, \dot{q}^i, \dot{p}_i) = (q^i, \dot{q}^i, \dot{p}_i, p_i).$$

Denote by α_V the 1-form on TT^*V defined as $\alpha_V = A_V^*(\theta_{TV})$. Locally we have

$$\alpha_V = \dot{p}_i\, \mathrm{d}q^i + p_i\, \mathrm{d}\dot{q}^i.$$

Then $(TT^*V, TV, T\pi_V, \alpha_V, A_V)$ is a special symplectic manifold (see Tulczyjew 1976b).

Remark. We notice that $\alpha_V \neq \beta_V$ but they differ by an exact 1-form, hence $\mathrm{d}\alpha_V = \mathrm{d}\beta_V$ and they define the same symplectic structure on TT^*V. To show this, we define a canonical function $\Omega_V: TT^*V \to R$ as follows:

$$\Omega_V(v) = \langle T\pi_V(v), \tau_{T^*V}(v)\rangle \qquad v \in TT^*V.$$

A simple computation in local coordinates shows that $\alpha_V - \beta_V = \mathrm{d}\Omega_V$.

Now, we assume that $L: TQ \to R$ is a *first-order Lagrangian* for a mechanical system. Consider the above construction for $V = Q$, $K = M = TQ$, $X = TT^*Q$. We can check that the Lagrangian submanifold defined by (2) is $N_2 = A_Q^{-1} \circ \mathrm{d}L(TQ)$, by taking into account (1). Then we have the following diagram:

$$\begin{tikzcd} N_2 \arrow[d, hook] & \\ TT^*Q \arrow[r, "A_Q"] \arrow[dr, "T\pi_Q"'] & T^*TQ \arrow[d, shift right, "\pi_{TQ}"'] \\ & TQ \arrow[u, shift right, "dL"'] \arrow[r, "L"] & R. \end{tikzcd}$$

In order to connect the Lagrangian with the Hamiltonian diagrams which exist independently, we need the Lagrangian L to be regular. Let us recall that L is said to be regular if the Hessian matrix of L with respect to the velocities $(\partial^2 L/\partial\dot{q}^i\partial\dot{q}^j)$ is non-singular, where $(q^i, \dot{q}^i)$ are coordinates for TQ. In such a case, the 2-form $\omega_L = \mathrm{d}\mathrm{d}_J L$, where J is the canonical *almost tangent structure* on TQ, is symplectic (see Godbillon 1969, de León and Rodrigues 1989). Then there exists a unique vector field ξ_L (called the *Lagrange vector field*) on TQ defined by the equation

$$i(\xi_L)\omega_L = -\mathrm{d}E_L \tag{4}$$

where $E_L = CL - L$ is the *energy* associated with L and C is the *Liouville vector field* on TQ. The projections onto Q of the integral curves of ξ_L satisfy the *Euler-Lagrange equations* for L:

$$(\partial L/\partial q^i) - (\mathrm{d}/\mathrm{d}t)(\partial L/\partial\dot{q}^i) = 0. \tag{5}$$

Let $\mathrm{Leg}: TQ \to T^*Q$ be the *Legendre transformation* for L. Then Leg is a local diffeomorphism locally given by

$$\mathrm{Leg}(q^i, \dot{q}^i) = (q^i, p_i)$$

where $p_i = (\partial L/\partial\dot{q}^i)$. Hence we have

$$\omega_L = (\mathrm{Leg})^*(\omega_Q).$$

Then the Lagrange vector field $\xi_L: TQ \to TTQ$ satisfies the relation

$$A_Q \circ T\,\mathrm{Leg} \circ \xi_L = \mathrm{d}L. \tag{6}$$

If Leg is a global diffeomorphism (i.e. L is *hyperregular*) then we can define the Hamiltonian function $H: T^*Q \to R$ as $H = E_L \circ (\mathrm{Leg})^{-1}$. Let ξ_H be the Hamiltonian vector field corresponding to H, i.e.

$$i_{\xi_H}\omega_Q = -\mathrm{d}H.$$

From (4)-(6), we easily check that

$$\xi_H = T\,\mathrm{Leg} \circ \xi_L \circ \mathrm{Leg}^{-1}$$

i.e. the following diagram:

$$\begin{tikzcd} TTQ \arrow[r, "T\mathrm{Leg}"] & TT^*Q \\ TQ \arrow[u, "\xi_L"] \arrow[r, "\mathrm{Leg}"] & T^*Q \arrow[u, "\xi_H"'] \end{tikzcd}$$

is commutative. Then we have

$$N_1 = \xi_H(T^*Q) = (\xi_H \circ \mathrm{Leg})(TQ) = (T\,\mathrm{Leg} \circ \xi_L)(TQ) = N_2.$$

We shall denote $N = N_1 = N_2$. Then we can construct the following diagram:

$$\begin{tikzcd}
 & & N \arrow[dll, bend right, "T\mathrm{Leg}"'] \arrow[d, hook] & \\
TTQ \arrow[dr, shift left, "\pi_{TQ}"] & T^*TQ \arrow[d, shift right] & TT^*Q \arrow[l, "A_Q"'] \arrow[r, "B_Q"] \arrow[dl, "T\pi_Q"'] \arrow[dr, shift right, "\tau_{T^*Q}"'] & T^*T^*Q \arrow[d, shift right, "\pi_{T^*Q}"'] \\
 & TQ \arrow[ul, shift left, "\xi_L"] \arrow[u, shift right, "\mathrm{d}L"'] \arrow[rr, "\mathrm{Leg}"] \arrow[dr, "L"'] & & T^*Q \arrow[ul, shift right, "\xi_H"'] \arrow[u, shift right, "-\mathrm{d}H"'] \arrow[dl, "H"] \\
 & & R &
\end{tikzcd}$$

We notice that the maps A_Q and B_Q are symplectomorphisms, establishing a one-to-one correspondence among the different Lagrangian submanifolds, i.e.

$$N = (A_Q^{-1} \circ \mathrm{d}L)(TQ) = [B_Q^{-1} \circ (-\mathrm{d}H)](T^*Q).$$

3. Time-independent higher-order Lagrangian and Hamiltonian dynamics

We first introduce a few new concepts. Given a manifold Q of dimension m, its tangent bundle T^kQ of order k is defined as the $((k+1)m)$-dimensional manifold of k-jets at 0 of maps from R into Q.

More precisely, given two such curves γ, γ' we say they are *kth-order tangent* at $q = \gamma(0) = \gamma'(0) \in Q$ if

$$(\mathrm{d}^r/\mathrm{d}t^r)(f(\gamma(t))(0) = (\mathrm{d}^r/\mathrm{d}t^r)(f(\gamma'(t))(0) \qquad \text{for } r = 1, 2, \ldots, k$$

for any real-valued function f defined in the neigbourhood of q. This defines an equivalence relation in the set of curves through q. The equivalence class or *k-jet* of γ at 0 is denoted by $j_0^k\gamma$.

If (q^i) are coordinates for Q then $(q_0^i, q_1^i, \ldots, q_k^i)$ are coordinates for T^kQ, where q_r^i, $0 \leqslant r \leqslant k$ is defined by

$$q_r^i(j_0^k\gamma) = (\mathrm{d}^r/\mathrm{d}t^r)(q^i(\gamma(t)))(0).$$

By $J_1(C_1)$ we denote the canonical *almost tangent structure of order k* (the *Liouville vector field*) on T^kQ. We set $J_r = (J_1)^r$, $1 \leqslant r$, and $C_r = J_{r-1}C_1$, $2 \leqslant r$. By d_T we denote a differential operator introduced by Tulczyjew (1975) which applies p-forms on T^kQ into p-forms on $T^{k+1}Q$.

A *Lagrangian of order k* is a function $L: T^kQ \to R$. We want to generalise to this case the description of first-order Lagrangian dynamics in § 2, but we will first review the constructions already known for this case (de León and Rodrigues 1985a, b).

We can construct a 2-form ω_L on $T^{2k-1}Q$ given by

$$\omega_L = \sum_{r=1}^{k} (-1)^{r-1}(1/r!)\, \mathrm{d}_T^{r-1}\, \mathrm{d}\mathrm{d}_{J_r}L.$$

Also, we define the *energy function* corresponding to L as the function E_L on $T^{2k-1}Q$ given by

$$E_L = \sum_{r=1}^{k} (-1)^{r-1}(1/r!)\, \mathrm{d}_T^{r-1}(C_r L) - L.$$

The Lagrangian L is said to be *regular* if the Hessian matrix

$$(\partial^2 L/\partial q_k^i \partial q_k^j)$$

is non-singular. One easily verifies that L is regular if and only if ω_L is symplectic. In such a case, there exists a unique vector field ξ_L (called a *Lagrange vector field*) on $T^{2k-1}Q$ such that

$$i(\xi_L)\omega_L = -\mathrm{d}E_L. \tag{7}$$

Then ξ_L is a *semispray* (or *2kth-order differential equation*), i.e. $J_1\xi_L = C_1$ and the projections onto Q of its integral curves are the solutions of the Euler-Lagrange equations for L:

$$\sum_{r=0}^{k} (-1)^r (\mathrm{d}^r/\mathrm{d}t^r)(\partial L/\partial q_r^i) = 0. \tag{8}$$

Finally, let us recall that the *Legendre transformation* is defined to be the map $\mathrm{Leg}: T^{2k-1}Q \to T^*(T^{k-1}Q)$ locally given by

$$\mathrm{Leg}(q_0^i, q_1^i, \ldots, q_{2k-1}^i) = (q_0^i, \ldots, q_{k-1}^i, p_i^0, \ldots, p_i^{k-1})$$

where p_i^r, $0 \leqslant r \leqslant k-1$ are the generalised momenta defined by

$$p_i^r = \sum_{s=0}^{k-r-1} (-1)^s\, \mathrm{d}_T^s(\partial L/\partial q_{r+s+1}^i).$$

If L is regular then Leg is a local diffeomorphism, and conversely.

There is a canonical inclusion $j: T^kQ \to T(T^{k-1}Q)$ defined by

$$j(j_0^k\gamma) = j_0^1\sigma$$

where $\sigma(t) = j_0^{k-1}\gamma_t$, $\gamma_t(s) = \gamma(s+t)$. We identify T^kQ with the submanifold $K = j(T^kQ)$ and Euler-Lagrange equations are globally defined by means of the Lagrangian submanifold generated by L in the symplectic manifold $TT^*(T^{k-1}Q)$ fibring onto $T(T^{k-1}Q)$.

Recall that in § 2 we defined a special symplectic structure for the map A_V. Applying this result to the case $V = T^{k-1}Q$, we get the following corollary.

Corollary. The quintuple

$$(TT^*(T^{k-1}Q), T(T^{k-1}Q), T\pi_{T^{k-1},Q}, \alpha_{T^{k-1}Q}, A_{T^{k-1}Q})$$

is a special symplectic manifold.

The generalisation of the diagram in § 2 is more complicated because $\mathrm{Leg}: T^{2k-1}Q \to$

$T^*(T^{k-1}Q)$ in this case and it is not defined in general in $T(T^{k-1}Q)$. The new diagram is:

$$\begin{tikzcd}
 & & & N \arrow[dd, shift left, hook] \arrow[dd, shift right, hook] & \\
 & N_1 \arrow[d, hook] & N_2 \arrow[d, hook] & & \\
TT^{2k-1}Q \arrow[rrr, bend left, "T\,\mathrm{Leg}"] & T^*T^{2k-1}Q \arrow[d, shift left, "\pi_{T^{2k-1}Q}"] & T^*T^*T^{k-1}Q \arrow[l, "T^*\mathrm{Leg}"'] \arrow[d, shift left, "\pi_{T^*T^{k-1}Q}"] & TT^*T^{k-1}Q \arrow[l, "B_{T^{k-1}Q}"'] \arrow[r, "A_{T^{k-1}Q}"] \arrow[d, "T\pi_{T^{k-1}Q}"] \arrow[dl, shift left, "\tau_{T^*T^{k-1}Q}"] & T^*TT^{k-1}Q \arrow[d, "j^*"] \\
 & T^{2k-1}Q \arrow[ul, "\xi_L"] \arrow[u, shift left, "-\mathrm{d}E_L"] \arrow[r, "\mathrm{Leg}"] \arrow[dr, "E_L"'] \arrow[ddr, "\pi_k^{2k-1}"'] & T^*T^{k-1}Q \arrow[u, shift left, "-\mathrm{d}H"] \arrow[ur, shift left, "\xi_H"] \arrow[d, "H"] & TT^{k-1}Q & T^*T^kQ \arrow[ddll, shift left, "\pi_{T^kQ}"] \\
 & & R & & \\
 & & T^kQ \arrow[u, "L"] \arrow[uur, "j"] \arrow[uurr, shift left, "\mathrm{d}L"] & &
\end{tikzcd}$$

Here T Leg and T^* Leg denote the tangent and cotangent maps associated with the Legendre transformation.

There are two different ways of inducing a Lagrangian submanifold from the given Lagrangian function L. One of them is by considering the inclusion j which permits us to consider L defined in the submanifold $K \subset TT^{k-1}Q$ in the base of the special symplectic manifold given in the corollary. This is always a non-trivial inclusion if $k \geqslant 2$, because of the different dimensions. This defines the Lagrangian submanifold N in the symplectic manifold $(TT^*T^{k-1}Q, \mathrm{d}\alpha_{T^{k-1}Q})$ from (2) by taking $X = TT^*T^{k-1}Q$, $M = TT^{k-1}Q$. The local expression of N is

$$\dot{p}_i^0 = (\partial L/\partial q^i) \qquad p_i^r = (\partial L/\partial q_{r+1}^i) - \dot{p}_i^{r+1} \qquad 0 \leqslant r \leqslant k-2$$
$$p_i^{k-1} = (\partial L/\partial q_k^i)$$

which give exactly the Euler-Langrange equations (8) for L (see de León and Lacomba 1988). From the proposition in § 2 we get the relation

$$j^*(A_{T^{k-1}Q}(N)) = \mathrm{d}L(T^kQ).$$

The other way is to pull back L to $T^{2k-1}Q$ via the projection $\pi_k^{2k-1}: T^{2k-1}Q \to T^kQ$ and then generate the Lagrangian submanifold N_1 in the symplectic manifold $(T^*T^{2k-1}Q, \omega_{T^{2k-1}Q})$ which is the image of $-\mathrm{d}E_L$, i.e.

$$N_1 = (-\mathrm{d}E_L)(T^{2k-1}Q).$$

Everything but the upper and left links in the square appearing on the left of the diagram is what we had in the case $k=1$. The commutative triangle in the upper right part of the diagram describes the Hamiltonian dynamics given a Hamiltonian function $H: T^*(T^{k-1}Q) \to R$. The corresponding Hamiltonian vector field is defined by

$$\xi_H = (B_{T^{k-1}Q})^{-1} \circ (-\mathrm{d}H)$$

which is equivalent to

$$i(\xi_H)\omega_{T^{k-1}Q} = -\mathrm{d}H$$

where $\omega_{T^{k-1}Q} = \mathrm{d}\theta_{T^{k-1}Q}$ is the canonical symplectic form on $T^*(T^{k-1}Q)$. Let N_2 be the Lagrangian submanifold in the symplectic manifold $(T^*T^*T^{k-1}Q, \omega_{T^*T^{k-1}Q})$ defined by

$$N_2 = (-\mathrm{d}H)(T^*T^{k-1}Q).$$

We consider below the link between the Lagrangian and the Hamiltonian formulations.

Assume that L is a regular Lagrangian, so that Leg is a (local) diffeomorphism. Then the (local) relation

$$T^* \,\mathrm{Leg} \circ \mathrm{d}H = \mathrm{d}E_L \circ \mathrm{Leg}^{-1}$$

is equivalent to the pull-back equation:

$$\mathrm{d}H = (\mathrm{Leg}^{-1})^*(\mathrm{d}E_L)$$

coming from $H = E_L \circ \mathrm{Leg}^{-1}$. It is not hard to show (de León and Rodriques 1985a, b) that the Lagrange vector field ξ_L is given by

$$\xi_L = (T\,\mathrm{Leg})^{-1} \circ \xi_H \circ \mathrm{Leg}$$

which is equivalent to (7), by using the definition of the Hamiltonian vector field ξ_H. Now we have

$$\begin{aligned}(T^*\,\mathrm{Leg})(N_2) &= (T^*\,\mathrm{Leg} \circ (-\mathrm{d}H))(T^*T^{k-1}Q)\\ &= ((-\mathrm{d}E_L) \circ \mathrm{Leg}^{-1})(T^*T^{k-1}Q) = (-\mathrm{d}E_L)(T^{2k-1}Q) = N_1.\end{aligned}$$

Furthermore, a simple computation in local coordinates shows that

$$B_{T^{k-1}Q}(N) = N_2.$$

Then the Lagrangian submanifolds N, N_1 and N_2 are taken to each other under the symplectomorphisms $B_{T^{k-1}Q}$ and T^* Leg at the top of the diagram.

4. Time-dependent first-order Lagrangian and Hamiltonian dynamics

Let $H: R \times T^*Q \to R$ be a time-dependent Hamiltonian. Then for each $t \in R$ fixed we set $H_t: T^*Q \to R$, $H_t(q^i, p_i) = H(t, q^i, p_i)$. We denote by ξ_{H_t} the Hamiltonian vector field corresponding to H_t, i.e.

$$i(\xi_{H_t})\omega_Q = -\mathrm{d}H_t.$$

Thus we construct a vector field ξ_H on $R \times T^*Q$ defined by

$$\xi_H(t, q^i, p_i) = (\partial/\partial t) + \xi_{H_t}(q^i, p_i).$$

Hence the integral curves of ξ_H satisfy the Hamilton equations (3) (see Abraham and Marsden 1978).

An alternative approach is the following (see Asorey *et al* 1983, Szebehely 1967, Thirring 1978, Weber 1985). We can extend H to a Hamiltonian $H^+: T^*(R \times Q) \to R$, where $T^*(R \times Q)$ is the extended phase space. H^+ is defined by

$$H^+(t, q^i, p^t, p_i) = H(t, q^i, p_i) + p^t$$

where (t, q^i, p^t, p_i) are local coordinates for $T^*(R \times Q)$.

Consider the case $V = R \times Q$, $K = M = T^*(R \times Q)$ in the construction of § 2, where

$$\beta_{R\times Q} = B^*_{R\times Q}(\theta_{T^*(R\times Q)}).$$

Then we obtain a Lagrangian submanifold N_1 in the symplectic manifold $(TT^*(R\times Q), \mathrm{d}\beta_{R\times Q})$ generated by $-H^+$.

As in § 2, we define

(*a*) $\xi_{H^+} = B^{-1}_{R\times Q} \circ \mathrm{d}(-H^+)$

and we have that

(*b*) the Hamilton equations for H^+ are

$$\dot{q}^i = (\partial H/\partial p_i) \qquad \dot{p}_i = -(\partial H/\partial q^i) \qquad \dot{p}^t = -(\partial H/\partial t)$$

(*c*) $N_1 = \xi_{H^+}(T^*(R\times Q))$.

If we denote by $\pi : T^*(R\times Q) \to R \times T^*Q$ the canonical projection defined by

$$\pi(t, q^i, p^t, p_i) = (t, q^i, p_i)$$

then a simple computation shows that the following diagram is commutative:

$$\begin{tikzcd}
R\times T^*Q \arrow[r, "\xi_H"] & T(R\times T^*Q) \\
T^*(R\times Q) \arrow[u, "\pi"] \arrow[r, "\xi_{H^+}"] & TT^*(R\times Q). \arrow[u, "T\pi"']
\end{tikzcd}$$

Moreover we have

$$T\pi(N_1) = \xi_H(R\times T^*Q).$$

Then we have the following diagram:

$$\begin{tikzcd}
N_1 \arrow[d, hook] & & & \\
TT^*(R\times Q) \arrow[r, "B_{R\times Q}"] \arrow[d, "T\pi"'] & T^*T^*(R\times Q) \arrow[d, shift right, "\pi_{T^*(R\times Q)}"'] & & \\
T(R\times T^*Q) & T^*(R\times Q) \arrow[ul, "\xi_{H^+}"'] \arrow[u, shift right, "-\mathrm{d}H^+"'] \arrow[dr, "H^+"] \arrow[dd, "\pi"'] & & T^*(R\times T^*Q). \\
 & & R & \\
 & R\times T^*Q \arrow[uul, "\xi_H"] \arrow[ur, "H"'] \arrow[uurr, "-\mathrm{d}H"'] & &
\end{tikzcd}$$

Let us remark that the triangular part connecting $T^*(R\times Q)$, $R\times T^*Q$ and R via π, H^+ and H is obviously not commutative.

Now, assume that $L : R\times TQ \to R$ is a *time-dependent first-order Lagrangian* for a mechanical system. Then we consider the canonical inclusion

$$i : R\times TQ \to T(R\times Q)$$

defined by

$$i(t, v) = (\partial/\partial t) + v.$$

Then i is locally given by

$$i(t, q^i, \dot{q}^i) = (t, q^i, 1, \dot{q}^i).$$

Consider the above construction in § 2 for $V = R \times Q$, $K = R \times TQ \subset M = T(R \times Q)$, $X = TT^*(R \times Q)$. Then the submanifold N_2 given by (2) is a Lagrangian submanifold in $(TT^*(R \times Q), \mathrm{d}\alpha_{R\times Q})$. If (t, q^i, p^t, p_i) are coordinates for $T^*(R \times Q)$, then N_2 is locally given by

$$\dot{p}^t = (\partial L/\partial t) \qquad \dot{p}_i = (\partial L/\partial q^i) \qquad p_i = (\partial L/\partial \dot{q}^i). \tag{9}$$

Clearly (9) gives the Euler–Lagrange equations for L.

We have the following diagram:

$$\begin{tikzcd}
N_2 \arrow[d, hook] & & \\
TT^*(R \times Q) \arrow[r, "A_{R\times Q}"] \arrow[dr, "T\pi_{R\times Q}"'] & T^*T(R \times Q) \arrow[d, "\pi_{T(R\times Q)}"] \arrow[ddr, "i^*"] & \\
& T(R \times Q) & \\
R & R \times TQ \arrow[l, "L"'] \arrow[u, "i"'] \arrow[r, "\mathrm{d}L"] & T^*(R \times TQ).
\end{tikzcd}$$

The proposition in § 2 is now

$$i^*(A_{R\times Q}(N_2)) = \mathrm{d}L(R \times TQ).$$

Now, assume that L is regular. Then the Legendre transformation for L is at least a local diffeomorphism

$$\mathrm{Leg}: R \times TQ \to R \times T^*Q$$

locally given by

$$\mathrm{Leg}(t, q^i, \dot{q}^i) = (t, q^i, p_i)$$

where $p_i = (\partial L/\partial \dot{q}^i)$. Define $H: R \times T^*Q \to R$ by $H = E_L \circ \mathrm{Leg}^{-1}$, where $E_L = CL - L$ is the energy associated with L, and put $H^+ = H + p^t$ as above. Then a simple computation in local coordinates show that $N_1 = N_2$. In the following we denote $N = N_1 = N_2$.

Then we consider the following diagram (which connects the Lagrangian and Hamiltonian diagrams):

$$\begin{tikzcd}
& N \arrow[d, hook] & \\
T^*T(R \times Q) \arrow[d, "i^*"'] & TT^*(R + Q) \arrow[l, "A_{R\times Q}"'] \arrow[r, "B_{R\times Q}"] \arrow[d, "T\pi"'] & T^*T^*(R \times Q) \\
T^*(R \times TQ) & T(R \times T^*Q) & T^*(R \times T^*Q) \\
& T(R \times TQ) \arrow[u, "T\mathrm{Leg}"] & \\
R \times TQ \arrow[uu, "\mathrm{d}L"] \arrow[ur, "\xi_L"'] \arrow[rr, "\mathrm{Leg}"] \arrow[dr, "L"'] \arrow[dd, "i"'] & & R \times T^*Q \arrow[uu, "-\mathrm{d}H"'] \arrow[uul, "\xi_H"] \arrow[dl, "H"'] \\
& R & \\
T(R \times Q) & & T^*(R \times Q). \arrow[uu, "\pi"'] \arrow[ul, "H^+"] \arrow[uuuuu, bend right, "-\mathrm{d}H^+"']
\end{tikzcd}$$

As above, let us remark that the triangular part connecting $T^*(R\times Q)$, $R\times T^*Q$ and R is not commutative, but we can easily check that

$$(T\,\mathrm{Leg}\circ\xi_L)(R\times TQ)=T\pi(N).$$

5. Time-dependent higher-order Lagrangian and Hamiltonian dynamics

Let $L\colon R\times T^kQ\to R$ be a *time-dependent Lagrangian of order k.* Then we consider the canonical inclusion

$$\begin{tikzcd} R\times T^kQ \arrow[r] \arrow[rr, bend left, "i"] & R\times TT^{k-1}Q \arrow[r] & T(R\times T^{k-1}Q) \end{tikzcd}$$

locally given by

$$i(t,q_0^i,q_1^i,\dots,q_k^i)=(t,q_0^i,\dots,q_{k-1}^i,1,q_1^i,\dots,q_k^i).$$

Consider the construction in § 2 for $V=R\times T^{k-1}Q$, $K=R\times T^kQ\subset M=T(R\times T^{k-1}Q)$, $X=TT^*(R\times T^{k-1}Q)$. Then the submanifold N_2 given by (2) is a Lagrangian submanifold of the symplectic manifold $(TT^*(R\times T^{k-1}Q),\ \mathrm{d}\alpha_{R\times T^{k-1}Q})$. If $(t,q_0^i,\dots,q_{k-1}^i,p^t,p_i^0,\dots,p_i^{k-1})$ are local coordinates for $T^*(R\times T^{k-1}Q)$, then N_2 is locally given by

$$\begin{aligned} &\dot p^t=(\partial L/\partial t)\\ &\dot p_i^0=(\partial L/\partial q^i) \qquad p_i^r=(\partial L/\partial q_{r+1}^i)-\dot p_i^{r+1} \qquad 0\leq r\leq k-2\\ &p_i^{k-1}=(\partial L/\partial q_k^i) \end{aligned}$$

which give the Euler–Lagrange equations for L.

We have the following diagram:

$$\begin{tikzcd} N_2 \arrow[d, hook] & & & \\ TT^*(R\times T^{k-1}Q) \arrow[rr, "A_{R\times T^{k-1}Q}"] \arrow[ddr, "T\pi_{R\times T^{k-1}Q}"'] & & T^*T(R\times T^{k-1}Q) \arrow[d, "\pi_{T(R\times T^{k-1}Q)}"] \arrow[ddr, "i^*"] & \\ & & T(R\times T^{k-1}Q) & \\ & R & R\times T^kQ \arrow[l, "L"'] \arrow[u, "i"] \arrow[r, "\mathrm{d}L"] & T^*(R\times T^kQ). \end{tikzcd}$$

Again from the proposition in § 2, we have that

$$i^*(A_{R\times T^{k-1}Q}(N_2))=\mathrm{d}L(R\times T^kQ).$$

Now, assume that L is regular. Then the Legendre transformation for L is at least a local diffeomorphism

$$\mathrm{Leg}\colon R\times T^{2k-1}Q\to R\times T^*(T^{k-1}Q)$$

given in coordinates by

$$\mathrm{Leg}(t,q_0^i,q_1^i,\dots,q_{2k-1}^i)=(t,q_0^i,\dots,q_{k-1}^i,p_i^0,\dots,p_i^{k-1}).$$

The energy E_L associated with L is now given by

$$E_L = \sum_{r=1}^{k} (-1)^r (1/r!)\, \overline{\mathrm{d}_T}(C_r L) - L$$

where $\overline{\mathrm{d}_T}$ is a differential operator which applies p-forms on $R \times T^k Q$ into p-forms on $R \times T^{k+1}$. In fact, $\overline{\mathrm{d}_T}$ is the natural extension of d_T involving derivatives with respect to the time (see de León and Rodrigues 1987). If we define $H : R \times T^*(T^{k-1}Q) \to R$ by $H = E_L \circ \mathrm{Leg}^{-1}$, then we obtain a Lagrangian submanifold N_1 in the symplectic manifold $(TT^*(R \times T^{k-1}Q), \mathrm{d}\beta_{R \times T^{k-1}Q})$ as in § 4 and we have

$$((B_{R \times T^{k-1}Q})^{-1} \circ (-\mathrm{d}H^+))(T^*(R \times T^{k-1}Q)) = N_1$$

where again $H^+ : T^*(R \times T^{k-1}Q) \to R$ is defined as $H^+ = H + p^t$ if p^t is the global momentum conjugate to time. A simple computation in local coordinates shows that $N_1 = N_2$ and we obtain a diagram similar to the last diagram of § 4.

Remark. We notice that all the constructions in this paper hold if the Lagrangians are degenerate, since given any H and any L we can generate corresponding Lagrangian submanifolds. These submanifolds may be completely unrelated in general. The same is true if, in addition, we consider constraints in the following sense: the Lagrangian L is defined on some submanifold of TQ or T^kQ, and so on.

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